

# Brane Induced Gravity in the Curved Bulk

Keiichi Akama

*Department of Physics, Saitama Medical University, Saitama, 350-0495, Japan*

Takashi Hattori

*Department of Physics, Kanagawa Dental College, Yokosuka, 238-8580, Japan*

(Dated: March 25, 2014)

Starting with the Nambu-Goto action of the braneworld embedded in a curved bulk, we derive the precise expressions for the quantum induced effects due to small fluctuations of the brane. To define the brane fluctuations invariantly, we introduce the Riemannian coordinate system for the subspace normal to the braneworld. It will turn out that we can systematically incorporate the effects of bulk curvature, and that the induced effects depend on the extrinsic curvature and the normal-connection gauge field as well as on bulk curvature components at the brane.

PACS numbers: 04.50.-h, 04.62.+v, 11.25.-w, 12.60.Rc

## I. INTRODUCTION

Recently, the ideas of braneworld and brane induced gravity [1]–[44] attracts much attention in wide areas of physics such as particle physics, field theory, superstring theory and cosmology. In the previous paper [44], we inquired their dynamical foundations and established the precise formulations for a rather limited setup of the flat bulk. We fixed imperfections in the naive old formalism. For example, the induced gravity terms are proportional to the number of the extra dimensions, but not to that of whole spacetime. We showed that the induced effects depend also on the extrinsic curvature and the normal-connection gauge field, and derived the precise expressions. Delicate situations for the cosmological-constant problem were discussed. Things were much simplified due to the flatness of bulk. In this paper, we extend the formulation to the fully general case of the curved bulk. It will turn out that we can incorporate the bulk curvature effects in a systematic way, as far as the small fluctuations are concerned. The induced effects depend also on the bulk curvature components at the brane, in addition to the extrinsic curvature and the normal-connection gauge field.

General relativity is based on the premises that the spacetime is curved affected by matter according to the Einstein equation, and that the objects move along the spacetime geodesics. The gravitations are apparent phenomena of the inertial motions in the curved spacetime. This successfully explains why the motion in the “gravitational” field is universal, i.e. blind to the object properties (mass, charge etc.), and why the Newton’s law of gravitation holds in daily familiar gravitational phenomena. The general relativity is supported by further precise observations such as perihelion precessions of planets, light bending due to massive stars etc.. It raises, however, another fundamental question why the spacetime is curved so as the Einstein equation indicates. It is supposed by Sakharov in 1967 that the quantum effects of the spacetime itself may provide the origin of the gravity with the Einstein equation [45]. Field theoretical

formulations were developed to realize the Einstein gravity as quantum effects of the matters [46]–[49]. In these theories, the gravitational field, i. e. spacetime metric appears as a composite of the matters.

On the other hand, the composite pictures of matters achieved great successes in wide areas of physics, including molecules, atoms, atomic nuclei, hadrons and various quasi-particles in quantum systems. Field theoretical treatments of particle compositeness have been explored extensively. According to Feynman rules, scattering amplitudes have a pole due to a particle in its energy variable at its mass. If a pole appears in some energy channel through some mechanism without a corresponding particle field in the original setup, it indicates existence of a composite particle formed by the mechanism. For example, contact interactions of fermions are known to give rise to a composite pole through chain diagrams, when we fix the momentum cut off at a large but finite level [50]. Some degrees of freedom at the short distances are converted into those of the collective modes. Such a composite is also successfully described by introducing an auxiliary field. If we eliminate the auxiliary field by using the constraint from the Euler equation in the original setup, we have a system described only with constituents. Its kinetic term is supplied through quantum loop diagrams. The methods are applied to, for example, superconductivity, models of hadrons, induced gauge theories [51], and models of composite quarks, leptons, gauge bosons, and Higgs bosons [52]. In renormalization theories (with large but finite momentum cut off), vanishing of the wave-function renormalization constant of some field implies absence of the kinetic term of the field in the original setup, despite its presence after renormalization. Then, the field is interpreted as a composite [53]. Vanishing of renormalization constants is called as compositeness condition. These methods have been extensively studied [54] and widely applied in condensed-matter, nuclear, and particle physics.

The field theoretical formulations of the induced gravity [46] is developed in the context of the unified composite model [52]. In the original setup, the system is

described only with matter fields with the assumption of general coordinate invariance. Or, equivalently, it is described with metric field which lacks the kinetic term, and is taken as an auxiliary field without independent degree of freedom. The kinetic term of the metric, i. e. the Einstein-Hilbert action, is induced via quantum loop diagrams, and the metric acquires the independent degrees of freedom. Thus, the metric is interpreted as the composite of the matters, and the origin of the gravitational phenomena is traced back to the quantum nature of the matters. Unfortunately, we can only partially apply the renormalization theory arguments because the general relativity is not renormalizable.

The simplest model of matters with general-coordinate invariance is that of scalar fields with the Nambu-Goto action [4]. Then, the scalar fields can be interpreted as the position coordinate of our spacetime in a spacetime with higher dimensions. Note that the spacetime itself is taken as a matter, which may induce quantum effects including gravity. This interpretation lead us to the ideas of the braneworld and the brane induced gravity [6], [11], [15]–[20]. These ideas have been studied extensively in these three decades. Physical models were constructed with topological defects in higher dimensional spacetime [6]–[18], and they were realized as “D-branes” in the superstring theory [21]–[24]. They were applied to the hierarchy problem with large extra dimensions, or with warped extra dimensions [22], [25]–[27]. It was argued that the brane induced gravity would imply various interesting consequences [28]–[30]. The ideas have been studied in wide areas including basic formalism, [31]–[35], brane induced gravity [36]–[37], particle physics phenomenology [38]–[40], and cosmology [41]–[43].

In this paper, we explore a precise formalism to derive the expressions for the quantum induced effects on the brane embedded in a curved bulk. For definiteness, we follow the simplest model of the braneworld with the Nambu-Goto dynamics. We do not specify dynamics of the bulk gravity, since it is irrelevant to the short distance effects at the brane. We treat the bulk curvature only as a given external field. To define the brane fluctuations invariantly, we introduce a geodesic measure (Riemannian coordinate) for the normal subspace to the braneworld. Then, we work out the quantum effects for the small fluctuation using the methods developed in the previous paper. It will turn out that we can systematically incorporate the effects of bulk curvature, and that the induced effects depend also on the bulk curvature components at the brane, as well as on the extrinsic curvature and the normal-connection gauge field. The plan of this paper is as follows. First, we define the model (Sec. II), and then we derive the quantum effects (Secs. III–VII). We define the brane fluctuations (Sec. III), formulate the quantum effects (Sec. IV), specify the method to regularize the divergences (Sec. V), classify the possible induced terms according to symmetries (Sec. VI), and calculate them via Feynman diagram method (Sec. VII). The final section (Sec. VIII) is devoted to conclusion and discussions.

The cosmological terms are fine-tuned, and the Einstein like gravity and other terms are induced.

## II. THE MODEL

We consider a quantum theoretical braneworld described by the Nambu-Goto Lagrangian [55]. We will see the quantum effects of the brane fluctuations give rise to effective braneworld gravity [6], [11], [15]–[20]. Let  $X^I(x^\mu)$  ( $I = 0, 1, \dots, D-1$ ) be the position of our three-brane in the  $D$  dimensional spacetime (bulk), parameterized by the brane coordinate  $x^\mu$  ( $\mu = 0, 1, 2, 3$ ), where  $I = 0$  and  $\mu = 0$  indicate the time components. Let  $G^{IJ}(X^K)$  be the bulk metric tensor at the bulk point  $X^K$ . This is taken to obey some bulk gravity theory. Then we consider a braneworld with dynamics given by the Nambu-Goto Lagrangian (density):

$$\mathcal{L}_{\text{br}} = -\lambda \sqrt{-\det_{\mu\nu} \left( \frac{\partial X^I}{\partial x^\mu} \frac{\partial X^J}{\partial x^\nu} G_{IJ}(X^K) \right)}, \quad (1)$$

where  $\lambda$  is a constant. Or we write it as

$$\mathcal{L}_{\text{br}} = -\lambda \sqrt{-g^{[X]}} \quad (2)$$

with abbreviations  $g^{[X]} = \det g_{\mu\nu}^{[X]}$ , and

$$g_{\mu\nu}^{[X]} = X^I_{,\mu} X^J_{,\nu} G_{IJ}(X^K), \quad (3)$$

where (and hereafter) indices following a comma (,) indicate differentiation with respect to the corresponding coordinate component, and  $[X]$  is attached to remind that they are abbreviations for expressions written in terms of  $X^I$ . The tensor  $g_{\mu\nu}^{[X]}$  is the induced metric on the brane with (3). We assume that  $X^I$  appears nowhere other than in  $\mathcal{L}_{\text{br}}$  in the total Lagrangian  $\mathcal{L}_{\text{tot}}$  including the bulk Lagrangian. The equation of motion from (1) is given by

$$g^{[X]\mu\nu} X_{;\mu\nu}^{[X]I} = 0, \quad (4)$$

where  $X_{;\mu\nu}^{[X]I}$  is the double covariant derivative with respect to both of the general coordinate transformations on the brane and to those in the bulk:

$$X_{;\mu\nu}^{[X]I} = X^I_{,\mu\nu} - X^I_{,\lambda} \gamma^{[X]\lambda}_{\mu\nu} + X^J_{,\mu} X^K_{,\nu} \Gamma^I_{JK} \quad (5)$$

with the affine connections on the brane and bulk

$$\gamma^{[X]\lambda}_{\mu\nu} = \frac{1}{2} g^{[X]\lambda\rho} \left( g_{\rho\mu,\nu}^{[X]} + g_{\rho\nu,\mu}^{[X]} - g_{\mu\nu,\rho}^{[X]} \right), \quad (6)$$

$$\Gamma^I_{JK} = \frac{1}{2} G^{IL} (G_{LJ,K} + G_{LK,J} - G_{JK,L}), \quad (7)$$

respectively. The system is invariant under the general coordinate transformation of the bulk and the brane separately. Under these symmetries, we can also have terms dependent on the curvature tensor written with  $g_{\mu\nu}^{[X]}$ .

They would, however, be suppressed for small curvatures as our exiting spacetime. Therefore we concentrate on the case where the Lagrangian is dominated by the simplest form (1). We expect that this gives a good approximation at low curvature limit in many dynamical models of the braneworld (e.g. topological defects [6]–[18], space-time singularities [22], [25]–[27], D-branes [21]–[24], etc.). It is remarkable that, as we shall see below, this simple model exhibits brane gravity and gauge theory like structure through the quantum effects.

For convenience of quantum treatments, we consider the following equivalent Lagrangian to (1):

$$\mathcal{L}'_{\text{br}} = -\frac{\lambda}{2}\sqrt{-g} [g^{\mu\nu} X^I_{,\mu} X^J_{,\nu} G_{IJ}(X^K) - 2] \quad (8)$$

where  $g_{\mu\nu}$  is an auxiliary field,  $g = \det g_{\mu\nu}$ , and  $g^{\mu\nu}$  is the inverse matrix of  $g_{\mu\nu}$ . Note that  $g_{\mu\nu}$ , unlike  $g^{[X]}_{\mu\nu}$  above, is treated as a field independent of  $X^I$ . Then the Euler Lagrange equations with respect to  $X^I$  and  $g_{\mu\nu}$  are given by

$$g^{\mu\nu} X^I_{;\mu\nu} = 0, \quad (9)$$

$$g_{\mu\nu} = X^I_{,\mu} X^J_{,\nu} G_{IJ}(X^K), \quad (10)$$

respectively, where the covariant derivative

$$X^I_{;\mu\nu} = X^I_{,\mu\nu} - X^I_{,\lambda} \gamma^\lambda_{\mu\nu} + X^J_{,\mu} X^K_{,\nu} \Gamma^I_{JK} \quad (11)$$

is written in terms of the brane affine connection

$$\gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) \quad (12)$$

with respect to the auxiliary field  $g_{\mu\nu}$ . Now  $g_{\mu\nu}$  in (9) is independent of  $X^I$ , and, instead, we have an extra equation (10), which guarantees that  $g_{\mu\nu}$  is the induced metric. If we substitute (10) into (9), we obtain the same equation as (4). Thus the systems with the Lagrangians  $\mathcal{L}_{\text{br}}$  and  $\mathcal{L}'_{\text{br}}$  coincide. Furthermore the argument that their Dirac bracket algebras coincide [47] indicates their quantum theoretical equivalence. We proceed hereafter based on the Lagrangian  $\mathcal{L}'_{\text{br}}$  instead of  $\mathcal{L}_{\text{br}}$ .

### III. BRANE FLUCTUATIONS

In order to extract the quantum effects of  $\mathcal{L}'_{\text{br}}$ , we deploy a semi-classical method, where we consider those due to small fluctuations of the brane around some classical solution (say  $Y^I(x^\mu)$ ) for  $X^I(x^\mu)$  of the equation of motion (4) [16]. Namely, the solution  $Y^I(x^\mu)$  obeys the classical equation

$$g^{\mu\nu} Y^I_{;\mu\nu} = 0, \quad (13)$$

$$g_{\mu\nu} = Y^I_{,\mu} Y^J_{,\nu} G_{IJ}(Y^K). \quad (14)$$

In quantum treatment,  $X^I$  itself in the Lagrangian  $\mathcal{L}'_{\text{br}}$  does not necessarily obey the equation of motion (4),

and may fluctuate from  $Y^I(x^\mu)$ . Among the fluctuations, only those transverse to the brane are physically meaningful, because those along the brane remain within the brane and cause no real fluctuations of the brane. They are absorbed by general coordinate transformations. In order to describe them, we choose  $D - 4$  independent normal vectors  $n_j^I(x^\mu)$  ( $j = 4, \dots, D - 1$ ) at each point on the brane with the orthogonality condition

$$n_j^I Y^J_{,\nu} G_{IJ}(Y^K) = 0. \quad (15)$$

We can arbitrarily choose a orthonormal system with

$$n_i^I n_j^J G_{IJ}(Y^K) = \eta_{ij} \equiv -\delta_{ij}, \quad (16)$$

where  $\delta_{ij}$  is the Kronecker delta. Then, we have the completeness relation of the vectors,

$$Y^I_{,\mu} Y^J_{,\nu} g^{\mu\nu} + n_i^I n_j^J \eta^{ij} = G^{IJ}(Y^K). \quad (17)$$

Throughout this paper, Latin capital suffices  $I, J, K, \dots$  indicate bulk coordinate indices running over the range  $0, 1, 2, \dots, D - 1$ , Greek lower case suffices  $\mu, \nu, \lambda, \dots$  indicate brane coordinate indices running over the range  $0, 1, 2, 3$ , and Latin lower case suffices  $i, j, k, \dots$  indicate extra-dimensional coordinate indices running over the range  $4, 5, \dots, D - 1$ . Bulk coordinate indices  $I, J, \dots (= 0, \dots, D - 1)$  are raised and lowered by the metric tensors  $G_{IJ}$  and  $G^{IJ}$ . We can read off from (8) and (10) that the auxiliary field  $g_{\mu\nu}$  plays the role the metric tensor on the brane. On the brane, we raise and lower the brane coordinate indices  $\mu, \nu, \dots (= 0, \dots, 3)$  by  $g_{\mu\nu}$  and  $g^{\mu\nu}$  (but not by  $g^{[X]}_{\mu\nu}$  and  $g^{[X]\mu\nu}$ ), and the normal space indices  $i, j, k, \dots (= 4, \dots, D - 1)$  by  $\eta_{ij}$  and  $\eta^{ij}$ .

In the previous paper, we assumed that the bulk is flat, and we defined the fluctuation measure  $\phi^i(x^\mu)$  with

$$X^I = Y^I + \phi^i n_i^I. \quad (18)$$

In this paper, we want to consider general cases where the bulk is also curved. Then, the definition of the fluctuation measure  $\phi^i(x^\mu)$  with (18) is inappropriate, because it relies on the coordinate system of  $X^I$ , and lacks the general-coordinate invariance of the bulk. We define the invariant measure  $\varphi^i(x^\mu)$  for quantum fluctuations of the brane as follows. Suppose that the geodesic curve in the direction of a normal unit vector  $n^I(x^\mu)$  of the solution brane  $Y^I$  hits the fluctuated brane  $X^I$  at a distance  $s$ . Then,  $\varphi^i$  is the coefficients of the expansion of  $sn^I$  in terms of  $n_i^I$ :

$$sn^I = \varphi^i n_i^I. \quad (19)$$

The position  $X^I$  of the fluctuated brane is given by

$$X^I = Y^I + \varphi^i n_i^I - \frac{1}{2} \Gamma^I_{JK} \varphi^i n_i^J \varphi^j n_j^K + \dots, \quad (20)$$

where  $\dots$  stands for terms of  $O((\varphi^i)^3)$  and the higher, and recursively given by solving the geodesic equation [56].

The higher terms of  $O((\varphi^i)^3)$  are unnecessary for our present purpose. If we take  $\varphi^i$  as independent variables but not functions of  $x^\mu$ , eq. (20) gives the transformation from the coordinate system  $(x^\mu, \varphi^i)$  to  $X^I$ . The former is called ‘‘Riemannian coordinate system’’. In the far regions off the brane in comparison with the scale of the brane curvature, this coordinate system may encounter singularities and multi-definitions. It causes, however, no problem, since only small fluctuations are necessary for our purpose. The higher terms in  $\varphi^i$ , however, may cause another problem in practical calculation of the quantum contributions. They give rise to the higher terms in  $\varphi^i$  in  $\mathcal{L}'_{\text{br}}$ , and hence the higher loop diagrams, which we have no systematic way to evaluate. In fact we do not even know how such large fluctuations contribute to the quantum effects. Here we restrict to retain the contributions only from the small fluctuations of the brane, and neglect the terms of  $O((\varphi^i)^3)$  and the higher in  $\mathcal{L}'_{\text{br}}$ . Accordingly, we are to evaluate only the one-loop diagrams.

Now we substitute (20) into the Lagrangian (8). For our purpose, it is sufficient to retain explicit forms for terms up to  $O((\varphi^i)^3)$ . Then, we obtain

$$\mathcal{L}'_{\text{br}} = \mathcal{L}'_0 + \mathcal{L}'_\varphi + O((\varphi^i)^3) \quad (21)$$

$$\mathcal{L}'_0 = -\frac{\lambda}{2}\sqrt{-g}(g^{\mu\nu}Y^I_{,\mu}Y^J_{,\nu}G_{IJ} - 2), \quad (22)$$

$$\mathcal{L}'_\varphi = -\frac{\lambda}{2}\sqrt{-g}[(D\varphi)_\mu{}^i(D\varphi)^\mu{}_i + \varphi^i\varphi^j Z_{ij}] \quad (23)$$

with

$$(D\varphi)_\mu{}^i = \varphi^i_{,\mu} + \varphi^k A^i_{k\mu}, \quad (24)$$

$$Z_{ij} = B_{i\mu\nu}B_j{}^{\mu\nu} + C_{ij} \quad (25)$$

where  $A_{ij\mu}$  and  $B_{i\mu\nu}$  are the normal connection and the extrinsic curvature, respectively, and  $C_{ij}$  is a particular combination of components of the bulk curvature tensor  $\mathcal{R}_{LIJK}$  at the brane in the Riemannian coordinate system. They are defined by

$$A_{ij\mu} = n_i{}^I n_j{}^J{}_{;\mu} G_{IJ}, \quad (26)$$

$$B_{i\mu\nu} = n_i{}^I Y^J{}_{;\mu\nu} G_{IJ}, \quad (27)$$

$$C_{ij} = g^{\mu\nu} Y^M{}_{,\mu} n_i{}^I n_j{}^J Y^N{}_{,\nu} \mathcal{R}_{MIJN}(Y^I), \quad (28)$$

where  $n_j{}^J{}_{;\mu}$  is the covariant derivative:

$$n_j{}^D{}_{;\mu} = n_n{}^D{}_{,\mu} + n_j{}^J Y^M{}_{,\mu} \Gamma^D_{JM}, \quad (29)$$

and the bulk curvature tensor is defined by

$$\mathcal{R}^L_{IJK} = \Gamma^L_{IK,J} - \Gamma^L_{IJ,K} + \Gamma^L_{TJ}\Gamma^T_{IK} - \Gamma^L_{TK}\Gamma^T_{IJ}. \quad (30)$$

We can see that (23) is the Lagrangian for the quantum scalar fields  $\varphi^m$  on the curved brane interacting with the given external fields  $A_{ij\mu}$ ,  $B_{i\mu\nu}$  and  $C_{ij}$ .

#### IV. QUANTUM EFFECTS

The quantum effects of the field  $\varphi^i$  are described by the effective Lagrangian  $\mathcal{L}^{\text{eff}}$

$$\int \mathcal{L}^{\text{eff}} d^4x = -i \ln \int [d\varphi^i] \exp \left[ i \int \mathcal{L}'_\varphi d^4x \right], \quad (31)$$

where  $[d\varphi^i]$  is the path-integration over  $\varphi^i$ . To perform it, we rewrite (23) into the form [57]

$$\mathcal{L}'_\varphi = -\frac{\lambda}{2} \varphi_i (-\delta^i_j \square + \mathcal{V}^i_j) \varphi^j, \quad (32)$$

with  $\square = \eta^{\mu\nu} \partial_\mu \partial_\nu$  and

$$\mathcal{V}^i_j = \delta^i_j \partial_\mu \mathcal{H}^{\mu\nu} \partial_\nu + \partial_\mu \mathcal{A}^i_{j\mu} + \mathcal{A}^i_{j\mu} \partial_\mu + \mathcal{Z}^i_j \quad (33)$$

$$\mathcal{H}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g} g^{\mu\nu} \quad (34)$$

$$\mathcal{A}^i_{j\mu} = -\sqrt{-g} A^i_{j\mu} \quad (35)$$

$$\mathcal{Z}^i_j = -\sqrt{-g} (A^{ik\mu} A_{kj\mu} - Z^i_j), \quad (36)$$

where the differential operator  $\partial_\mu \equiv \partial/\partial x^\mu$  is taken to operate on the whole expression in its right side in (32). The path-integration in (31) is performed to give

$$\int \mathcal{L}^{\text{eff}} d^4x = \sum_{n=0}^{\infty} \frac{1}{2ni} \text{Tr} \left( \frac{1}{\square} \mathcal{V}^j_k \right)^n, \quad (37)$$

up to additional constants, where Tr indicates the trace over the brane coordinate variable  $x^\mu$  and extra dimension index  $j$ . The terms in (37) can be calculated with Feynman-diagram method. In terms of the Fourier transforms

$$\tilde{\mathcal{H}}^{\mu\nu}(q_l) = \int d^4x \mathcal{H}^{\mu\nu}(x) e^{iq_l x}, \quad (38)$$

$$\tilde{\mathcal{A}}^i_{j\mu}(q_l) = \int d^4x \mathcal{A}^i_{j\mu}(x) e^{iq_l x}, \quad (39)$$

$$\tilde{\mathcal{Z}}^j_k(q_l) = \int d^4x \mathcal{Z}^j_k(x) e^{iq_l x}, \quad (40)$$

the effective Lagrangian  $\mathcal{L}^{\text{eff}}$  is written as

$$\mathcal{L}^{\text{eff}} = \sum_{n=0}^{\infty} \frac{1}{2n} \prod_{l=1}^n \int \frac{d^4 q_l}{(2\pi)^4} e^{-iq_l x} G^n, \quad (41)$$

$$G^n = \int \frac{d^4 p}{i(2\pi)^4} \prod_{l=1}^n \frac{1}{-p_l^2} \tilde{\mathcal{V}}^{k_l}_{k_{l-1}}(p_l, q_l), \quad (42)$$

$$\tilde{\mathcal{V}}^{k'}_{k'}(p_l, q_l) = -\delta^{k'}_{k'}(p_l)_\mu (p_{l-1})_\nu \tilde{\mathcal{H}}^{\mu\nu}(q_l) - i(p_l + p_{l-1})_\mu \tilde{\mathcal{A}}^{k'}_{k'}{}^\mu(q_l) + \tilde{\mathcal{Z}}^{k'}_{k'}(q_l), \quad (43)$$

where  $p_l = p + q_1 + \dots + q_l$  and  $k_0 = k_n$ . The function  $G^n$  is nothing but the Feynman amplitude for the one-loop diagram with  $n$  internal lines of  $\varphi^j$  and  $n$  vertices of  $\tilde{\mathcal{V}}^{k'}_{k'}$  (FIG. 1). Unfortunately, the  $p$ -dependence of the integrand in (41) with (42) indicates that the integration over

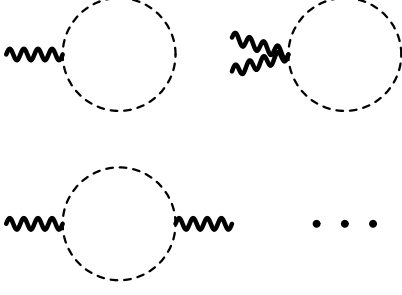


FIG. 1: The Feynman diagrams. The dashed lines indicate the  $\varphi^i$ -propagators, and the wavy lines indicate external fields of  $A_{ij\mu}$ ,  $B_{i\mu\nu}$  or  $h_{\mu\nu}$ . The dots indicate an infinite series of diagrams with a dashed-line loop and with more external wavy lines than two. By virtue of the symmetries of the system, we have only to calculate the three diagrams explicitly drawn here.

$p$  diverges at most quartically. The divergences will be regulated in the next section. Then, we can perform the integration over  $p$  to obtain the function  $G^n$ . The  $q_l$ 's are replaced by differentiation  $i\partial_l$  of the  $l$ -th vertex function according to the inverse Fourier transformation in (41). Collecting all the contributions, which are functions of the fields  $g_{\mu\nu}$ ,  $A_{ij\mu}$  and  $B_{i\mu\nu}$  and their derivatives, we can obtain the expression for the effective Lagrangian  $\mathcal{L}^{\text{eff}}$ .

## V. DIVERGENCES AND REGULARIZATION

The  $p$ -dependence of the integrand in (41) with (42) indicates that the integration over  $p$  diverges at most quartically. We expect, however, that fluctuations with smaller wave length than the brane thickness are suppressed. Then, the momenta higher than the inverse of the thickness are cut off. In order to model the cut-off without violating full symmetry of  $\mathcal{L}'_{\text{br}}$ , we introduce three Pauli-Villars regulators  $\Phi_r^m$  with very large mass  $M_r$  ( $r=1,2,3$ ), which are taken equal finally,  $M_r \rightarrow \Lambda$ , following the method of the original paper [4]. Precisely, it amounts to consider the regularized effective Lagrangian

$$\mathcal{L}^{\text{reg}} = \mathcal{L}^{\text{eff}} + \sum_{r=1}^3 C_r \mathcal{L}_{M_r}^{\text{eff}} \quad (44)$$

where  $\mathcal{L}_{M_r}^{\text{eff}}$  is the effective Lagrangian for the quantum effects from  $\mathcal{L}'_{\Phi_r}$  which is the same as  $\mathcal{L}'_{\varphi}$  except that  $\varphi^i$  is replaced by the regulator field  $\Phi_r^i$  with mass  $M_r$ ,

$$\int \mathcal{L}_{M_r}^{\text{eff}} d^4x = -i \ln \int [d\Phi_r^i] \exp \left[ i \int \mathcal{L}'_{\Phi_r} d^4x \right], \quad (45)$$

$$\mathcal{L}'_{\Phi_r} = \mathcal{L}'_{\varphi}|_{\varphi=\Phi_r} + \frac{1}{2} \lambda M_r^2 \sqrt{-g} \Phi_r^i \Phi_r^j \eta_{ij}. \quad (46)$$

In (44), the coefficients  $C_r$  are defined by the coupled algebraic equation

$$\sum_{r=1}^3 C_r = -1, \quad \sum_{r=1}^3 C_r (M_r)^2 = \sum_{r=1}^3 C_r (M_r)^4 = 0. \quad (47)$$

Note that the added mass term also preserves the full symmetry of  $\mathcal{L}'_{\text{br}}$ . Performing the path integration over  $\Phi_r^i$ , we have

$$\int \mathcal{L}_{M_r}^{\text{eff}} d^4x = \sum_{n=0}^{\infty} \frac{1}{2ni} \text{Tr} \left( \frac{1}{\square + M_r^2} \mathcal{V}_{M_r}{}^k{}_{k'} \right)^n, \quad (48)$$

$$\mathcal{V}_{M_r}{}^k{}_{k'} = \mathcal{V}^k{}_{k'} + \mathcal{F} M_r^2 \delta_{k'}^k, \quad (49)$$

$$\mathcal{F} = 1 - \sqrt{-g}, \quad (50)$$

with  $\mathcal{V}^k{}_{k'}$  in (33). In terms of the Fourier transform

$$\tilde{\mathcal{F}}(q_l) = \int d^4x \mathcal{F}(x) e^{iq_l x}. \quad (51)$$

we have

$$\mathcal{L}_{M_r}^{\text{eff}} = \sum_{n=0}^{\infty} \frac{1}{2n} \prod_{l=1}^n \int \frac{d^4 q_l}{(2\pi)^4} e^{-iq_l x} G_{M_r}^n, \quad (52)$$

$$G_{M_r}^n = \int \frac{d^4 p}{i(2\pi)^4} \prod_{l=1}^n \frac{1}{-p_l^2 + M_r^2} \tilde{\mathcal{V}}_{M_r}{}^{k_l}{}_{k_{l-1}}(p_l, q_l), \quad (53)$$

$$\tilde{\mathcal{V}}_{M_r}{}^k{}_{k'}(p_l, q_l) = \tilde{\mathcal{V}}^k{}_{k'}(p_l, q_l) + \delta_{k'}^k M_r^2 \tilde{\mathcal{F}}(q_l), \quad (54)$$

with  $\tilde{\mathcal{V}}^k{}_{k'}(p_l, q_l)$  in (43). In dimensional regularization, the divergent parts of the Feynman amplitude  $G_{M_r}^n$  behaves like

$$G_{M_r}^n \sim \epsilon^{-1} (\mathcal{G}_4 M_r^4 + \mathcal{G}_2 M_r^2 + \mathcal{G}_0), \quad (55)$$

where this is evaluated at the spacetime dimension  $4-2\epsilon$ , and  $\mathcal{G}_{2v}$  are the appropriate coefficient functions. The singularities at  $\epsilon = 0$  reflect the divergences in the  $p$ -integration. We can see that, when they are summed with the coefficients  $C_r$  over  $r$  in (44), they cancel out according to (47). Therefore, the  $p$ -integrations in  $\mathcal{L}^{\text{reg}}$  converge. Any positive power contributions of  $M_r$  regular at infinity vanish according to (47). The function  $G_{M_r}^n$  involves logarithmic singularities in  $M_r$ . In the equal mass limit  $M_r \rightarrow \Lambda$ ,

$$\sum_{r=1}^3 C_r M_r^4 \ln M_r^2 \rightarrow -\Lambda^4/2 \quad (56)$$

$$\sum_{r=1}^3 C_r M_r^2 \ln M_r^2 \rightarrow \Lambda^2/2 \quad (57)$$

$$\sum_{r=1}^3 C_r \ln M_r^2 \rightarrow -\ln \Lambda^2 \quad (58)$$

## VI. CLASSIFICATION OF THE TERMS

Thus the divergent part  $\mathcal{L}^{\text{div}}$  of the regularized effective Lagrangian  $\mathcal{L}^{\text{reg}}$  consists of the terms which are proportional to  $\Lambda^4$ ,  $\Lambda^2$  or  $\ln \Lambda^2$ , and are monomials of  $\mathcal{H}^{\mu\nu}$ ,  $\mathcal{F}$ ,  $\mathcal{A}^i_{j\mu}$ ,  $\mathcal{Z}^i_j$ , and their derivatives. The expressions  $\mathcal{H}^{\mu\nu}$ ,  $\mathcal{F}$ ,  $\mathcal{A}^i_{j\mu}$ , and  $\mathcal{Z}^i_j$  are written in terms of the fields  $g_{\mu\nu}$ ,  $A^i_{j\mu}$ , and  $Z^i_j$  according to (34), (50), (35), and (36). Introducing the notation  $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ , we rewrite  $g^{\mu\nu}$  and  $\sqrt{-g}$  in  $\mathcal{H}^{\mu\nu}$ ,  $\mathcal{F}$ ,  $\mathcal{A}^i_{j\mu}$ , and  $\mathcal{Z}^i_j$  according to

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\nu}_{(2)} + h^{\mu\nu}_{(3)} + \dots, \quad (59)$$

$$\sqrt{-g} = 1 + h/2 - h_{(2)}/4 + h^2/8 + \dots, \quad (60)$$

with [58]

$$h_{(n)}^{\mu\nu} = \overbrace{h^\mu_\sigma h^\sigma_\tau \dots h^\rho_\nu}^n, \quad (61)$$

$$h = h^\mu_\mu, \quad h_{(n)} = h_{(n)}^\mu{}_\mu. \quad (62)$$

Then,  $\mathcal{L}^{\text{div}}$  becomes an infinite sum of monomials of  $h_{\mu\nu}$ ,  $A^i_{j\mu}$ ,  $Z^i_j$ , and their derivatives. Let us denote the numbers of  $h_{\mu\nu}$ ,  $A^i_{j\mu}$ ,  $Z^i_j$ , and the differential operators in the monomial by  $N_h$ ,  $N_A$ ,  $N_Z$  and  $N_\partial$ , respectively. The Lagrangian  $\mathcal{L}^{\text{reg}}$  should have mass dimension 4, while  $h_{\mu\nu}$ ,  $A^i_{j\mu}$ ,  $Z^i_j$ , and the differential operator has mass dimension 0, 1, 2, and 1, respectively. Therefore, the numbers  $N_A$ ,  $N_Z$  and  $N_\partial$  are restricted by

$$N_A + 2N_Z + N_\partial \leq 4 - 2k_{\text{div}}, \quad (63)$$

where  $k_{\text{div}} = 2, 1, 0$  for  $\Lambda^4$ ,  $\Lambda^2$ , and  $\ln \Lambda^2$  terms, respectively. On the other hand, the number  $N_h$  of  $h_{\mu\nu}$  is not restricted. The relation (63) allows only finite numbers of values of  $N_A$ ,  $N_Z$  and  $N_\partial$ , according to which we can classify the terms of  $\mathcal{L}^{\text{div}}$ . Each class involves infinitely many terms for arbitrary values of  $N_h$ .

They are, however, not all independent, because they are related by high symmetry of the system under the general coordinate transformations on the brane and  $\text{SO}(4)$  gauge transformations of the normal space rotation. Owing to the symmetry of the system, only finite number of terms are allowed. The general coordinate transformation symmetry requires that the effective Lagrangian density is proportional to  $\sqrt{-g}$  times a sum of invariant forms. We list the allowed invariant forms in TABLE I, where  $R = R^\mu{}_\mu$ ,  $R_{\mu\nu} = R^\lambda{}_{\mu\nu\lambda}$ , and

$$R^\kappa{}_{\lambda\mu\nu} = \gamma^\kappa_{\lambda\nu,\mu} - \gamma^\kappa_{\lambda\mu,\nu} + \gamma^\kappa_{\rho\mu}\gamma^\rho_{\lambda\nu} - \gamma^\kappa_{\rho\nu}\gamma^\rho_{\lambda\mu}, \quad (64)$$

$$A_{ij\mu\nu} = A_{ij\nu,\mu} - A_{ij\mu,\nu} + A_{ik\mu}A^k_{j\nu} - A_{ik\nu}A^k_{j\mu}, \quad (65)$$

is the field strength of the gauge field  $A_{ij\mu}$ . Among the invariants,  $R^2$ ,  $R_{\mu\nu}R^{\mu\nu}$ , and  $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$  are not all independent, but related by Gauss-Bonnet relation.

## VII. CALCULATION

Thus we can calculate the coefficients of the term  $\sqrt{-g}$  times the invariant forms by calculating the lowest order

TABLE I: Invariant forms

$k^{\text{div}}$	$N_A$	$N_Z$	$N_\partial$	invariant forms
2	0	0	0	1
1	0	0	2	$R$
	0	1	0	$Z^i_i$
0	0	0	4	$R^2$ , $R_{\mu\nu}R^{\mu\nu}$ , $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$
	0	1	2	$RZ^i_i$ , $Z^i_{i;\mu}{}^\mu$
	0	2	0	$(Z^i_i)^2$ , $Z^{ij}Z_{ij}$
	2,3,4	0	$4 - N_A$	$A_{ij\mu\nu}A^{ij\mu\nu}$

contributions in  $h_{\mu\nu}$ . The lowest contributions to the term with  $N_A = N_Z = N_\partial = 0$  are  $\mathcal{O}(h_{\mu\nu})$ , while those to  $N_A = N_Z = 0$  and  $N_\partial \neq 0$  are  $\mathcal{O}((h_{\mu\nu})^2)$ , because the  $\mathcal{O}(h_{\mu\nu})$  terms are total derivatives. Therefore, their lowest terms are in the one- and two-point functions  $G^1$  and  $G^2$ . The only possible form including  $A_{ij\mu}$  is  $A_{ij\mu\nu}A^{ij\mu\nu}$  and its lowest term is of  $\mathcal{O}((h_{\mu\nu})^0)$ , and it is in  $G^2$ . Thus, it is sufficient to calculate  $G^1$  and  $G^2$  in order to determine full contributions to  $\mathcal{L}^{\text{div}}$ .

From (53), (54) and (43), they are given by

$$G^1_{M_r} = -N_{\text{ex}}\tilde{\mathcal{H}}^{\mu\nu}I_{\mu\nu} + N_{\text{ex}}M_r^2\tilde{\mathcal{F}}I + \tilde{\mathcal{Z}}^i_i I, \quad (66)$$

$$G^2_{M_r} = N_{\text{ex}}\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{H}}^{\lambda\rho}J_{\mu\nu\lambda\rho} + 2i\tilde{\mathcal{H}}^{\mu\nu}\tilde{\mathcal{A}}^i_{i\rho}J_{\mu\nu\rho} - (N_{\text{ex}}M_r^2\tilde{\mathcal{F}} + \tilde{\mathcal{Z}}^i_i)\tilde{\mathcal{H}}^{\mu\nu}(J_{\mu\nu} - q_\mu q_\nu J)/4 - \tilde{\mathcal{A}}^i_{j\mu}\tilde{\mathcal{A}}^j_{i\nu}J_{\mu\nu} - 2i(M_r^2\tilde{\mathcal{F}}\delta^i_j + \tilde{\mathcal{Z}}^i_j)\tilde{\mathcal{A}}^j_{i\rho}J_\mu + (N_{\text{ex}}M_r^4\tilde{\mathcal{F}}\tilde{\mathcal{F}} + M_r^2\tilde{\mathcal{F}}\tilde{\mathcal{Z}}^i_i + \tilde{\mathcal{Z}}^i_j\tilde{\mathcal{Z}}^j_i)J, \quad (67)$$

where  $N_{\text{ex}} = D - 4$  is the number of the extra dimensions,  $q_\mu$  is the momentum flowing in and out through the vertices, and

$$I = \int \frac{d^4p}{i(2\pi)^4} \frac{1}{[-p^2 + M_r^2]}, \quad (68)$$

$$I_{\mu\nu} = \int \frac{d^4p}{i(2\pi)^4} \frac{p_\mu p_\nu}{[-p^2 + M_r^2]}, \quad (69)$$

$$J = \int \frac{d^4p}{i(2\pi)^4} \frac{1}{[-(p+q)^2 + M_r^2][ -p^2 + M_r^2]}, \quad (70)$$

$$J_\mu = \int \frac{d^4p}{i(2\pi)^4} \frac{(2p+q)_\mu}{[-(p+q)^2 + M_r^2][ -p^2 + M_r^2]}, \quad (71)$$

$$J_{\mu\nu} = \int \frac{d^4p}{i(2\pi)^4} \frac{(2p+q)_\mu(2p+q)_\nu}{[-(p+q)^2 + M_r^2][ -p^2 + M_r^2]}, \quad (72)$$

$$J_{\mu\nu\rho} = \int \frac{d^4p}{i(2\pi)^4} \frac{(p+q)_\mu p_\nu (2p+q)_\rho}{[-(p+q)^2 + M_r^2][ -p^2 + M_r^2]}, \quad (73)$$

$$J_{\mu\nu\lambda\rho} = \int \frac{d^4p}{i(2\pi)^4} \frac{(p+q)_\mu p_\nu p_\lambda (p+q)_\rho}{[-(p+q)^2 + M_r^2][ -p^2 + M_r^2]}. \quad (74)$$

In the dimensional regularization, for large  $M_r^2$ , they are calculated to be

$$I = \frac{M_r^{2-2\epsilon}}{16\pi^2\epsilon}, \quad J = \frac{M_r^{-2\epsilon}}{16\pi^2\epsilon}, \quad (75)$$

$$J_\mu = 0, \quad J_{\mu\nu\rho} = 0, \quad (76)$$

$$I_{\mu\nu} = -\frac{M_r^{2-2\epsilon}}{16\pi^2\epsilon}\eta_{\mu\nu}, \quad (77)$$

$$J_{\mu\nu} = -\frac{M_r^{2-2\epsilon}}{16\pi^2\epsilon} \left[ 2M_r^2\eta_{\mu\nu} + \frac{1}{3}(q_\mu q_\nu - q^2\eta_{\mu\nu}) \right], \quad (78)$$

$$J_{\mu\nu\lambda\rho} = \frac{M_r^{2-2\epsilon}}{16\pi^2\epsilon} \left[ \left( \frac{M_r^4}{8} - \frac{M_r^2 q^2}{24} + \frac{q^4}{240} \right) S_{\mu\nu\lambda\rho} - \left( \frac{M_r^2}{12} - \frac{q^2}{60} \right) T_{\mu\nu\lambda\rho} + \left( \frac{M_r^2}{6} - \frac{q^2}{40} \right) T'_{\mu\nu\lambda\rho} + \frac{1}{30} q_\mu q_\nu q_\lambda q_\rho \right] \quad (79)$$

with

$$S_{\mu\nu\lambda\rho} = \eta_{\mu\nu}\eta_{\lambda\rho} + \eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda}, \quad (80)$$

$$T_{\mu\nu\lambda\rho} = \eta_{\mu\nu}q_\lambda q_\rho + \eta_{\mu\rho}q_\lambda q_\nu + \eta_{\nu\lambda}q_\mu q_\rho + \eta_{\lambda\rho}q_\mu q_\nu, \quad (81)$$

$$T'_{\mu\nu\lambda\rho} = \eta_{\mu\lambda}q_\nu q_\rho + \eta_{\nu\rho}q_\mu q_\lambda. \quad (82)$$

We substitute (75)–(79) into (66) and (67), and substitute them into (41) to get  $\mathcal{L}^{\text{eff}}$ , and rearrange the terms into a sum of monomials of  $h_{\mu\nu}$ ,  $A^i_{j\mu}$ ,  $Z^i_j$  and their derivatives. The divergent terms are proportional to  $M_r^{2v-2\epsilon}/\epsilon$  ( $v = 0, 1, 2$ ). When summed over  $r = 1, 2, 3$  in the regularization (44), the singularities of  $1/\epsilon$  in  $\mathcal{L}^{\text{reg}}$  cancel out owing to (47). Then, in the limit of  $\epsilon \rightarrow 0$ , there remain the factors  $M_r^{2v} \ln M_r^2$ , and, in the equal mass limit  $M_r \rightarrow \Lambda$ , eqs. (56)–(58) indicate that

$$\sum_r C_r M_r^{-2\epsilon}/\epsilon \rightarrow \ln \Lambda^2, \quad (83)$$

$$\sum_r C_r M_r^{2-2\epsilon}/\epsilon \rightarrow -\Lambda^2/2, \quad (84)$$

$$\sum_r C_r M_r^{4-2\epsilon}/\epsilon \rightarrow \Lambda^4/2, \quad (85)$$

The terms are classified as follows.

(i) The terms with  $N_A = N_Z = 0$  are given by [58]

$$\begin{aligned} & \frac{N_{\text{ex}}}{32(4\pi)^2} \left[ \frac{\Lambda^4}{2} (4h - 2h_{(2)} + h^2) \right. \\ & + \frac{\Lambda^2}{3} (h^{\mu\nu,\lambda} h_{\mu\nu,\lambda} - 2h^{\mu\nu}{}_{,\nu} h_\mu{}^\lambda{}_{,\lambda} + 2h^{\mu\nu}{}_{,\nu} h_{,\mu}{}^\lambda - h^\mu{}_{,\mu} h_{,\mu}) \\ & + \frac{\ln \Lambda^2}{15} \{ h^{\mu\nu,\lambda\rho} h_{\mu\nu,\lambda\rho} - 2h^{\mu\nu}{}_{,\nu\rho} h_\mu{}^\lambda{}_{,\lambda\rho} \\ & \quad \left. + 4(h^{\mu\nu}{}_{,\mu\nu})^2 - 6h^{\mu\nu}{}_{,\mu\nu} h^\lambda{}_{,\lambda} + 3(h^\mu{}_{,\mu})^2 \} \right] \quad (86) \end{aligned}$$

up to total derivatives. Because the full expression should have the symmetry, they should be the lower order expression of  $\sqrt{-g}$  times the invariant forms in table I. The terms in (86) are to be compared with the lower contributions for  $\sqrt{-g}$  in (60) and

$$\sqrt{-g}R = \frac{1}{4} (h^{\mu\nu,\lambda} h_{\mu\nu,\lambda} - 2h^{\mu\nu}{}_{,\nu} h_\mu{}^\lambda{}_{,\lambda} + 2h^{\mu\nu}{}_{,\nu} h_{,\mu}{}^\lambda - h^\mu{}_{,\mu} h_{,\mu}), \quad (87)$$

$$\sqrt{-g}R^2 = (h^{\mu\nu}{}_{,\mu\nu})^2 - 2h^{\mu\nu}{}_{,\mu\nu} h^\lambda{}_{,\lambda} + (h^\mu{}_{,\mu})^2, \quad (88)$$

$$\sqrt{-g}R_{\mu\nu}R^{\mu\nu} = -\frac{1}{4} [h^{\mu\nu,\lambda\rho} h_{\mu\nu,\lambda\rho} - 2h^{\mu\nu}{}_{,\nu\rho} h_\mu{}^\lambda{}_{,\lambda\rho} + 2(h^{\mu\nu}{}_{,\mu\nu})^2 - 2h^{\mu\nu}{}_{,\mu\nu} h^\lambda{}_{,\lambda} + (h^\mu{}_{,\mu})^2], \quad (89)$$

where total derivatives are neglected. Note that we have changed the sign convention of the curvature tensor

$R^\kappa_{\lambda\mu\nu}$  (64) from that of the previous paper [44].

(ii) The lowest contributions to  $\mathcal{L}^{\text{reg}}$  with  $N_Z \neq 0$  and  $N_\partial = 0$  are

$$\frac{1}{4(4\pi)^2} (\Lambda^2 Z^i{}_i + \ln \Lambda^2 Z^{ij} Z_{ij}), \quad (90)$$

which are taken as the lowest parts of the forms  $\sqrt{-g}Z^i{}_i$  and  $\sqrt{-g}Z^{ij}Z_{ij}$ .

(iii) The lowest contribution with  $N_Z \neq 0$  and  $N_\partial = 2$  is

$$\frac{\ln \Lambda^2}{12(4\pi)^2} (h^{\mu\nu}{}_{,\mu\nu} + h^\mu{}_{,\mu}) Z^i{}_i \quad (91)$$

which is the lowest part of the form  $\sqrt{-g}RZ^i{}_i$ .

(iv) The lowest contribution with  $N_A \neq 0$  is

$$\frac{\ln \Lambda^2}{24(4\pi)^2} (A_{ij\mu,\nu} - A_{ij\nu,\mu})(A^{ij\mu,\nu} - A^{ij\nu,\mu}), \quad (92)$$

which is the lowest part of the form  $\sqrt{-g}A_{ij\mu\nu}A^{ij\mu\nu}$  with  $N_A = 2$ . Note that it suffices to determine the coefficient of the form in  $\mathcal{L}^{\text{reg}}$ .

Collecting the results of (i)–(iv), we finally obtain the expression for the divergent part  $\mathcal{L}^{\text{div}}$  of  $\mathcal{L}^{\text{reg}}$ :

$$\begin{aligned} \mathcal{L}^{\text{div}} = & \frac{\sqrt{-g}}{(4\pi)^2} \left[ N_{\text{ex}} \left( \frac{\Lambda^4}{8} + \frac{\Lambda^2}{24} R + \frac{\ln \Lambda^2}{240} (R^2 + 2R_{\mu\nu}R^{\mu\nu}) \right) \right. \\ & + \frac{\Lambda^2}{4} Z^i{}_i + \frac{\ln \Lambda^2}{4} Z_{ij}Z^{ij} + \frac{\ln \Lambda^2}{12} RZ^i{}_i \\ & \left. - \frac{\ln \Lambda^2}{24} A_{ij\mu\nu}A^{ij\mu\nu} \right]. \quad (93) \end{aligned}$$

In terms of the fields  $A_{ij\mu}$ ,  $B_{i\mu\nu}$  and  $C_{ij}$ , (93) is rewritten as

$$\begin{aligned} \mathcal{L}^{\text{div}} = & \frac{\sqrt{-g}}{(4\pi)^2} \left[ N_{\text{ex}} \left( \frac{\Lambda^4}{8} + \frac{\Lambda^2}{24} R + \frac{\ln \Lambda^2}{240} (R^2 + 2R_{\mu\nu}R^{\mu\nu}) \right) \right. \\ & + \frac{\Lambda^2}{4} B_{i\mu\nu}B^{i\mu\nu} + \frac{\ln \Lambda^2}{4} B_{i\mu\nu}B_j{}^{\mu\nu}B^i{}_{\lambda\rho}B^{j\lambda\rho} \\ & + \frac{\ln \Lambda^2}{12} R B_{i\mu\nu}B^{i\mu\nu} - \frac{\ln \Lambda^2}{24} A_{ij\mu\nu}A^{ij\mu\nu} \\ & + \frac{\Lambda^2}{4} C^i{}_i + \frac{\ln \Lambda^2}{4} C_{ij}C^{ij} \\ & \left. + \frac{\ln \Lambda^2}{12} R C^i{}_i + \frac{\ln \Lambda^2}{2} B_{i\mu\nu}B_j{}^{\mu\nu}C^{ij} \right], \quad (94) \end{aligned}$$

where  $A_{ij\mu\nu}$  is the field strength (65) of the normal-connection gauge field  $A_{ij\mu}$  (26). The divergences cannot be renormalized because the original action does not have these terms. They are, however, cut off by the momentum cut off at the inverse of the brane thickness. The fluctuations with smaller wave length than the brane thickness make no sense, and do not contribute to the quantum loop effects. Therefore, the contributions in (94) give rise to genuine quantum induced effects.

### VIII. CONCLUSIONS AND DISCUSSIONS

We have established a precise formalism to derive the expressions for the quantum induced effects due to small fluctuations of the braneworld embedded in a curved bulk. Assuming general coordinate invariance both of the brane and of the bulk, we adopted the simplest model with the Nambu-Goto action (1) of the brane. In the previous paper [44], we inquired this problem for the limited case with a flat bulk. In this paper, we extended it to the fully general case where the bulk is arbitrarily curved. To define the brane fluctuations in the curved bulk invariantly, we introduced the Riemannian coordinate  $\varphi^m$  (20) for the normal geodesic subspace of the braneworld. Then, we worked out the quantum effects of the small fluctuations using the methods developed in the previous paper. It turned out that we can systematically incorporate the effects of bulk curvature, and that the induced effects depend also on the bulk curvature components at the brane, in addition to the extrinsic curvature and the normal-connection gauge field. The resultant expression for the induced effects is given by (94).

The first three lines in the right hand side of (94) have the same form as the result (88) with flat bulk [44], while the last two lines explicitly depend on the bulk curvature components. In the first line, the first term in the big curly bracket contributes a huge amount to the cosmological term, and suffers from the notorious problem of naturalness, as was discussed in the previous paper [44]. It requires unnatural fine-tuning among the parameters and the spacetime configurations, though it is possible. If the cosmological term is successfully suppressed in total, the terms with the brane curvature in the big curly bracket provide the kinetic term of the metric  $g_{\mu\nu}$ , i. e. the gravity is effectively induced. For small curvatures, it is dominated by the Einstein-Hilbert action (the second term in the big curly bracket) with small corrections from the terms quadratic in the curvature (the last term in the big curly bracket). An approximate Einstein equation takes place effectively. The sign of the term is right to give ordinary attractive gravity in accordance with the observation. Its magnitude indicates that the cutoff  $\Lambda$ , and hence, the brane thickness is order of the Planck scale.

The second line in (94) includes the mass term (the first term) of the extrinsic curvature  $B_{i\mu\nu}$  and its self-interaction terms (the second term). The field  $B_{i\mu\nu}$  has no kinetic term and a kind of massive background field which does not propagate. The mass is order of the Planck scale and the self-interactions are suppressed compared with its mass. The first term in the third line gives rise to an interaction of  $B_{i\mu\nu}$  with  $R^\kappa_{\mu\nu\lambda}$ , and hence, with  $g_{\mu\nu}$  through it. The interactions are also suppressed compared with its mass. The second term in the third line is nothing but the gauge invariant kinetic term of the gauge field  $A_{ij\mu}$ . This term includes the kinetic term and the self-interaction terms of  $A_{ij\mu}$ . The field  $A_{ij\mu}$  looks like a composite gauge field. This has the gauge symmetry of

the rotational group  $SO(N_{\text{ex}})$  of the normal space to the braneworld. The magnitude coupling constant is order of  $1/\sqrt{N_{\text{ex}} \ln \Lambda^2}$ . This can be a candidate of the origin of existing gauge symmetries in the particle theories [15]. The fields  $A_{ij\mu}$  and  $B_{i\mu\nu}$  interact also with  $g_{\mu\nu}$  through the factors  $\sqrt{-g}$  and  $g^{\mu\nu}$  which is the inverse of  $g_{\mu\nu}$ . In summary, the terms in the second and the third lines describe the behaviors of the fields  $A_{ij\mu}$  and  $B_{i\mu\nu}$  interacting with the gravitational fields  $g_{\mu\nu}$  induced from the terms in the first line.

Though these terms have the same forms as the result (88) with flat bulk [44], the fields  $g_{\mu\nu}$ ,  $A_{ij\mu}$  and  $B_{i\mu\nu}$  are related to the bulk curvature  $\mathcal{R}^L_{IJK}$  at the brane, through the Gauss-Codazzi-Ricci equations:

$$R_{\rho\lambda\mu\nu} + B_{i\rho\nu}B^i_{\lambda\mu} - B_{i\rho\mu}B^i_{\lambda\nu} = \bar{\mathcal{R}}_{\rho\lambda\mu\nu}, \quad (95)$$

$$B_{i\lambda[\nu,\mu]} + A_{ij\mu}B^j_{\lambda\nu} - A_{ij\nu}B^j_{\lambda\mu} = \bar{\mathcal{R}}_{i\lambda\mu\nu}, \quad (96)$$

$$A_{ij\mu\nu} + B_{i\nu\rho}B_{j\mu}{}^\rho - B_{i\mu\rho}B_{j\nu}{}^\rho = \bar{\mathcal{R}}_{ij\mu\nu}, \quad (97)$$

where  $\bar{\mathcal{R}}_{\rho\lambda\mu\nu}$ ,  $\bar{\mathcal{R}}_{i\lambda\mu\nu}$  and  $\bar{\mathcal{R}}_{ij\mu\nu}$  are particular components of the bulk curvature tensor  $\mathcal{R}_{IJKL}$  at the brane in the Riemannian-coordinate:

$$\bar{\mathcal{R}}_{\rho\lambda\mu\nu} = Y^I_{,\rho} Y^J_{,\lambda} Y^K_{,\mu} Y^L_{,\nu} \mathcal{R}_{IJKL}(Y^I), \quad (98)$$

$$\bar{\mathcal{R}}_{i\lambda\mu\nu} = n_i{}^I Y^J_{,\lambda} Y^K_{,\mu} Y^L_{,\nu} \mathcal{R}_{IJKL}(Y^I), \quad (99)$$

$$\bar{\mathcal{R}}_{ij\mu\nu} = n_i{}^I n_j{}^J Y^K_{,\mu} Y^L_{,\nu} \mathcal{R}_{IJKL}(Y^I). \quad (100)$$

If the bulk curvature were determined by some condition in the bulk physics, the configurations of the fields  $g_{\mu\nu}$ ,  $A_{ij\mu}$  and  $B_{i\mu\nu}$  should be restricted by the Gauss-Codazzi-Ricci equations (95)–(97). In fact, they are the integrability condition for the embedding of the brane. Unfortunately, it may be very difficult to know the physical conditions to determine the bulk curvature.

On the other hand, the last two lines (the forth and the last) in (94) explicitly depend on the bulk curvature at the braneworld through  $C_{ij}$  in (28), which is rewritten as  $C_{ij} = g^{\mu\nu} \bar{\mathcal{R}}_{\mu ij\nu}$  with

$$\bar{\mathcal{R}}_{\mu ij\nu} = Y^L_{,\mu} n_i{}^I n_j{}^J Y^K_{,\nu} \mathcal{R}_{LIJK}(Y^I). \quad (101)$$

They are absent in the flat-bulk results [44]. Note that the components in (101) are different from those in (98)–(100) which appear in the Gauss-Codazzi-Ricci equations. The field  $C_{ij}$  is entirely independent of the fields  $g_{\mu\nu}$ ,  $A_{ij\mu}$  and  $B_{i\mu\nu}$ . We can see this (independence) also from the fact that, in the definition of the bulk curvature (30), the components in (98)–(100) do not include the differentiation of the bulk metric with respect to the normal coordinate, while (101) does. The forth line of (94) describes behavior of the field  $C_{ij}$  without kinetic term, indicating that it is a background field without propagation. The last line describes interactions of the field  $C_{ij}$  with the scalar curvature  $R$  of the brane and those with the extrinsic curvature  $B_{i\mu\nu}$ . The field  $C_{ij}$  interacts also with  $g_{\mu\nu}$  through the factor  $\sqrt{-g}$ . In summary, the terms in the forth and the last lines describe the behaviors of the field  $C_{ij}$  interacting with the gravitational fields  $g_{\mu\nu}$  induced from the terms in the first line.



## Acknowledgments

We would like to thank Professor G. R. Dvali, Professor G. Gabadadze, Professor M. E. Shaposhnikov, Professor I. Antoniadis, Professor M. Giovannini, Professor S. Randjbar-Daemi, Professor R. Gregory, Professor P. Kanti, Professor G. Gibbons, Professor K. Hashimoto, Professor E. J. Copeland, Professor D. L. Wiltshire, Professor I. P. Neupane, Professor R. R. Volkas, Professor A. Kobakhidze, Professor C. Wetterich, Pro-

fessor M. Shifman, Professor A. Vainshtein, Professor D. Wands, Professor M. Visser, Professor T. Inami, Professor I. Oda and Professor H. Mukaida for invaluable discussions and their kind hospitality extended to us during our stay in their places.

This work was supported by Grant-in-Aid for Scientific Research, No. 13640297, 17500601, and 22500819 from Japanese Ministry of Education, Culture, Sports, Science and Technology.

- 
- [1] C. Fronsdal, *Nuovo Cim.* **13**, 988 (1959).
  - [2] D. W. Joseph, *Phys. Rev.* **126**, 319 (1962).
  - [3] T. Regge and C. Teitelboim, in *Proceedings, Marcel Grossmann Meeting On General Relativity*, (Oxford, 1977) p. 77-87.
  - [4] K. Akama, *Prog. Theor. Phys.* **60**, 1900 (1978).
  - [5] M. D. Maia, *Rev. Bras. Fis.* **8**, 429 (1978).
  - [6] K. Akama, *Lect. Notes Phys.* **176**, 267 (1982).
  - [7] V. A. Rubakov and M. E. Shaposhnikov, *Phys. Lett. B* **125**, 136 (1983).
  - [8] M. D. Maia and W. Mecklenburg, *J. Math. Phys.* **25**, 3047 (1984).
  - [9] H. Nicolai and C. Wetterich, *Phys. Lett. B* **150**, 347 (1985).
  - [10] M. Visser, *Phys. Lett. B* **159**, 22 (1985).
  - [11] M. Pavšič, *Class. Quant. Grav.* **2**, 869 (1985); *Phys. Lett. A* **107**, 66 (1985).
  - [12] C. G. Callan Jr. and J. A. Harvey, *Nucl. Phys. B* **250**, 427 (1985).
  - [13] J. Hughes, J. Liu and J. Polchinski, *Phys. Lett. B* **180**, 370 (1986).
  - [14] G. W. Gibbons and D. L. Wiltshire, *Nucl. Phys. B* **287**, 717 (1987).
  - [15] K. Akama, *Prog. Theor. Phys.* **78**, 184 (1987).
  - [16] I. Watanabe, *Bulletin of the College of Liberal Arts and Sciences, Tokyo Medical and Dental University*, **17**, 1-17 (1987).
  - [17] K. Akama, *Prog. Theor. Phys.* **79**, 1299 (1988).
  - [18] K. Akama, *Prog. Theor. Phys.* **80**, 935 (1988).
  - [19] A. Nakamura, S. Hirenzaki and K. Shiraishi, *Nucl. Phys. B* **339**, 533 (1990).
  - [20] M. Pavsic, *Found. Phys.* **24**, 1495 (1994); *Grav. Cosmol.* **2**, 1 (1996).
  - [21] J. Dai, R. G. Leigh and J. Polchinski, *Mod. Phys. Lett. A* **4**, 2073 (1989).
  - [22] Ignatios Antoniadis, *Phys. Lett. B* **246**, 377 (1990).
  - [23] J. Polchinski, *Phys. Rev. Lett.* **75**, 4724-4727 (1995).
  - [24] P. Horava and E. Witten, *Nucl. Phys. B* **460**, 506-524 (1996); *Nucl. Phys. B* **475**, 94-114 (1996).
  - [25] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *Phys. Lett. B* **429**, 263-272 (1998); *Phys. Rev. D* **59**, 086004 (1999).
  - [26] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, *Phys. Lett. B* **436**, 257-263 (1998).
  - [27] L. Randall, R. Sundrum, *Phys. Rev. Lett.* **83**, 3370-3373 (1999); **83**, 4690-4693 (1999).
  - [28] R. Gregory, V. A. Rubakov and S. M. Sibiryakov, *Phys. Rev. Lett.* **84**, 5928 (2000).
  - [29] G. Dvali, G. Gabadadze, and M. Porrati, *Phys. Lett. B* **485**, 208 (2000).
  - [30] G. Dvali and G. Gabadadze, *Phys. Rev. D* **63**, 065007 (2001).
  - [31] Ichiro Oda, *Phys. Lett. B* **496**, 113 (2000); G. R. Dvali, S. Randjbar-Daemi and R. Tabbash, *Phys. Rev. D* **65**, 064021 (2002); B. M. N. Carter, A. B. Nielsen, and D. L. Wiltshire, *JHEP* **0607**, 034 (2006); Y. Kodama, K. Kokubu and N. Sawado, *Phys. Rev. D* **78**, 045001 (2008); *Phys. Rev. D* **79**, 065024 (2009); A. Davidson, D. P. George, A. Kobakhidze, R. R. Volkas, K. C. Wali, *Phys. Rev. D* **77**, 085031 (2008).
  - [32] N. A. Butcher and P. M. Saffin, *JHEP* **0711**, 062 (2007); M. C. B. Abdalla, M. E. X. Guimaraes and J. M. Hoff da Silva, *Eur. Phys. J. C* **55**, 337 (2008); V. V. Khrushchov, *Grav. Cosmol.* **13**, 259 (2007); F. Coradeschi, S. De Curtis, D. Dominici and J. R. Pelaez, *JHEP* **0804**, 048 (2008); C. Wetterich, *Phys. Rev. D* **77**, 103505 (2008). *D* **78**, 043503 (2008); *D* **81**, 103508 (2010); *Phys. Rev. Lett.* **102**, 141303 (2009).
  - [33] B. P. Kosyakov, *Int. J. Mod. Phys. A* **23**, 4695 (2008); D. Maity, *Phys. Rev. D* **78**, 084008 (2008); R. Gregory, *Lect. Notes Phys.* **769**, 259 (2009); Y. X. Liu, Z. H. Zhao, S. W. Wei and Y. S. Duan, *JCAP* **0902**, 003 (2009).
  - [34] K. Akama, T. Hattori and H. Mukaida, *Mod. Phys. Lett. A* **26**, 2869 (2011); arXiv:1109.0840; arXiv:1208.3303.
  - [35] K. -i. Maeda and K. Uzawa, *Phys. Rev. D* **85**, 086004 (2012); S. Guha and P. Bhattacharya, arXiv:1207.5716; Y. -X. Liu, K. Yang, H. Guo and Y. Zhong, *Phys. Rev. D* **85**, 124053 (2012); A. Halavanau, T. Romanczukiewicz and Y. Shnir, *Phys. Rev. D* **86**, 085027 (2012); C. Adam, J. M. Queiruga, J. Sanchez-Guillen and A. Wereszczynski, *Phys. Rev. D* **86**, 105009 (2012).
  - [36] K. Akama and T. Hattori, *Mod. Phys. Lett. A* **15**, 2017 (2000); N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, *JHEP* **0405**, 074 (2004); C. Charmousis, R. Gregory, N. Kaloper, and A. Padilla, *JHEP* **0610**, 066 (2006); K. Akama, arXiv: gr-qc/0607106. N. Kaloper, *Mod. Phys. Lett. A* **23**, 781 (2008); G. Gabadadze, *Nucl. Phys. Proc. Suppl.* **171**, 88 (2007).
  - [37] A. Krause, *Towards Dark Energy from String-Theory, in Proceedings of 6th International Heidelberg Conference on Dark matter in astroparticle and particle physics (DARK 2007), Sydney, Australia (World Scientific, 2007) p.181-197; M. Cadoni and P. Pani, Phys. Lett. B* **674**, 308 (2009); M. R. Setare, *Int. J. Mod. Phys. D* **18**, 419 (2009); K. Nozari, M. RSetare, T. Azizi and S. Akhshabi, *Acta Phys. Polon. B* **41**, 897 (2010); M. Heydari-Fard and

- H. R. Sepangi, JCAP **0901**, 034 (2009); K. Atazadeh, A. M. Ghezelbash and H. R. Sepangi, Int. J. Mod. Phys. D **21**, 1250069 (2012).
- [38] D. Ida, K. Y. Oda and S. C. Park, Phys. Rev. D **67**, 064025 (2003) [Erratum-ibid. D **69**, 049901 (2004)]; P. Kanti, Int. J. Mod. Phys. A **19**, 4899 (2004); J. Phys. Conf. Ser. **189**, 012020 (2009).
- [39] B. Korutlu, arXiv: 0801.3579; G. Landsberg, Proceedings, 13th Lomonosov Conference on Elementary Particle Physics, Moscow, Russia, August 23-29 (Moscow State U., 2008) p.99-108; T. E. Clark, S. T. Love, M. Nitta, T. ter Veldhuis and C. Xiong, Nucl. Phys. B **810** (2009) 97; T. E. Clark, S. T. Love, C. Xiong, M. Nitta and T. ter Veldhuis, Phys. Rev. D **78**, 115004 (2008).
- [40] M. Sarrazin, F. Petit, Phys. Rev. D **83**, 035009 (2011); Phys. Rev. D **81** 035014 (2010); M. Sarrazin, G. Pignol, F. Petit and V. V. Nesvizhevsky, Phys. Lett. B **712**, 213 (2012); N. Garrido and H. H. Hernandez, arXiv:1201.3951; Y. Fujimoto, T. Nagasawa, K. Nishiwaki and M. Sakamoto, PTEP 2013 023B07 (2013).
- [41] P. Kanti, I. I. Kogan, K. A. Olive and M. Pospelov, Phys. Lett. B **468**, 31 (1999); P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **565**, 269 (2000); T. Shiromizu, K. I. Maeda, and M. Sasaki, Phys. Rev. D **62**, 024012 (2000); R. Maartens, D. Wands, B. A. Bassett and I.P.C. Heard, Phys. Rev. D **62**, 041301(R) (2000); J. Garriga and T. Tanaka, Phys. Rev. Lett. **84**, 2778 (2000); K. Koyama and J. Soda, Phys. Rev. D **62**, 123502 (2000); E. J. Copeland, A. R. Liddle and J. E. Lidsey, Phys. Rev. D **64**, 023509 (2001); R. Maartens, Living Rev. Rel. **7**, 7 (2004).
- [42] K. Koyama, Gen. Rel. Grav. **40**, 421 (2008); M. Heydari-Fard and H. R. Sepangi, Phys. Rev. D **76**, 104009 (2007); E. N. Saridakis, Nucl. Phys. B **808**, 224 (2009); C. Adam, N. Grandi, J. Sanchez-Guillen and A. Wereszczynski, J. Phys. A **41**, 212004 (2008) [Erratum-ibid. A **42**, 159801 (2009)]; K. Atazadeh, M. Farhoudi and H. R. Sepangi, Phys. Lett. B **660**, 275 (2008).
- [43] I. C. Jardim, R. R. Landim, G. Alencar and R. N. Costa Filho, Phys. Rev. D **84**, 064019 (2011); D. Maity, arXiv:1209.0862 Phys. Rev. D **86** 084056 (2012); L. J. S. Sousa, C. A. S. Silva and C. A. S. Almeida, Phys. Lett. B **718** 579-583 (2012).
- [44] K. Akama and T. Hattori, Class. Quant. Grav. **30**, 205002 (2013) [arXiv:1309.3090 [gr-qc]].
- [45] A. D. Sakharov, Dokl. Akad. Nauk SSSR **177**, 70 (1967) [Sov. Phys. Dokl. **12**, 1040 (1968)].
- [46] K. Akama, Y. Chikashige and T. Matsuki, Prog. Theor. Phys. **59**, 653 (1978); K. Akama, Y. Chikashige, T. Matsuki and H. Terazawa, Prog. Theor. Phys. **60**, 868 (1978); K. Akama, Ref [4]; A. Zee, Phys. Rev. Lett. **42**, 417 (1979); S. L. Adler, Phys. Rev. Lett. **44**, 1567 (1980); K. Akama, Phys. Rev. D **24**, 3073 (1981);
- [47] K. Akama, Prog. Theor. Phys. **61**, 687 (1979);
- [48] C. Barcelo, S. Liberati and M. Visser, Class. Quant. Grav. **18**, 3595 (2001); C. Barcelo, M. Visser and S. Liberati, Int. J. Mod. Phys. D **10**, 799 (2001); C. Barcelo, S. Liberati and M. Visser, Living Rev. Rel. **8**, 12 (2005); Living Rev. Rel. **4**, 3 (2011).
- [49] B. Broda and M. Szanecki, Phys. Lett. B **674**, 64 (2009); C. Wetterich, Phys. Lett. B **704**, 612 (2011); Phys. Rev. D **85**, 104017 (2012); Lect. Notes Phys. **863**, 67-92 (2013); Annals Phys. **327**, 2184 (2012); D. Sexty and C. Wetterich, Nucl. Phys. B **867** 290-329 (2013).
- [50] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345.
- [51] J. D. Bjorken, Ann. Phys. **24** (1963) 174.
- [52] H. Terazawa, Y. Chikashige and K. Akama, Phys. Rev. D **15**, 480 (1977).
- [53] B. Jouvet, Nuovo Cim. **5** 1133 (1956); M. T. Vaughn, R. Aaron and R. D. Amado, Phys. Rev. **124**, 1258 (1961); A. Salam, Nuovo Cim. **25** 224 (1962); S. Weinberg, Phys. Rev. **130** 776 (1963).
- [54] E. Tomboulis, Phys. Lett. B **70**, 361 (1977); B **97**, 77 (1980); K. I. Shizuya, Phys. Rev. D **21**, 2327 (1980); K. Akama, Prog. Theor. Phys. **64** 1494 (1980); **84** 1212 (1990); K. Akama, Phys. Rev. Lett. **76**, 184 (1996); Nucl. Phys. A **629**, 37C (1998); Phys. Lett. B **583**, 207 (2004); K. Akama and T. Hattori, Phys. Lett. B **392**, 383 (1997); **445**, 106 (1998); Phys. Rev. Lett. **93**, 211602 (2004); A. Akabane and K. Akama, Prog. Theor. Phys. **112**, 757 (2004).
- [55] Y. Nambu, Duality And Hadrodynamics, Copenhagen High-Energy Summer Symposium, Aug. 1970, in Broken Symmetry, p. 280-301 (World Scientific, 1995); T. Goto, Prog. Theor. Phys. **46**, 1560 (1971); Y. Nambu, Phys. Rev. D **10**, 4262 (1974).
- [56] L. P. Eisenhart, "Riemannian Geometry" (Princeton University Press, 1925), p.52.
- [57] Eq. (27) in the previous paper [44] includes a typographic error. Correctly, its right hand side should have an extra minus sign in front of it.
- [58] Note that  $h_{\mu\nu}$  and its derivatives are not brane tensors, and their superscripts are not defined by multiplying  $g^{\mu\nu}$ . For convenience of calculations, we define here that their subscripts are raised by  $\eta^{\mu\nu}$ .